

LOOSE LEGENDRIAN EMBEDDINGS IN HIGH DIMENSIONAL CONTACT MANIFOLDS

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ABSTRACT. We give an h-principle type result for a class of Legendrian embeddings in contact manifolds of dimension at least 5. These Legendrians, referred to as loose, have trivial pseudo-holomorphic invariants. We demonstrate they are classified up to ambient contact isotopy by smooth embedding class equipped with an almost complex framing. This result is inherently high dimensional: analogous results in dimension 3 are false.

1. INTRODUCTION

Let (Y^{2n+1}, ξ) be a contact manifold. A *Legendrian knot* is defined to be a closed, connected, embedded submanifold $L^n \rightarrow Y$ so that $TL \subseteq \xi$. Though we abuse notation and say $L \subseteq Y$, we study parametrized embeddings everywhere in the paper. Legendrian knots of particular interest include knots with topology S^n , and/or knots embedded in $(\mathbb{R}^{2n+1}, \xi_{std} = \ker(dz - \sum y_i dx_i))$.

In this paper we prove an h-principle result for a class of Legendrian knots. To state the theorem we first must define formal Legendrian knots, which can be thought of as knots which “look Legendrian to an algebraic topologist”.

Definition 1.1. Let $f : L^n \hookrightarrow (Y^{2n+1}, \xi)$ be a smooth embedding, and let $F_s : TL \rightarrow TY|_L$ be a homotopy of bundle monomorphisms covering f for all s , so that $F_0 = df$ and $F_1(TL)$ is a Lagrangian subspace of ξ . (Recall ξ has a canonical conformal symplectic structure.) The pair (f, F_s) is called a *formal Legendrian knot*.

A Legendrian knot can be thought of as a formal Legendrian by letting $F_s = df$ for all s . In particular, we say that two Legendrian knots are *formally isotopic* if there exists a smooth isotopy $f_t : L \rightarrow Y$ between them, and df_t is homotopic through paths of monomorphisms, fixed at the endpoints, to a path of Lagrangian monomorphisms. Notice that classifying formal Legendrian knots up to formal isotopy is a question purely about smooth topology and bundle theory, we do this for the case $(Y, \xi) = \mathbb{R}_{std}^{2n+1}$ in Appendix A. There are many infinite families of distinct knots which are formally isotopic which can be distinguished with pseudo-holomorphic curve invariants [8].

Informally, we call a Legendrian knot with $n > 1$ *loose* if it contains a sufficiently thick Weinstein neighborhood of a stabilized Legendrian curve; we give a precise

definition in the following section. The main purpose of this paper is a proof of

Theorem 1.2. *Suppose $n > 1$ and fix a contact manifold (Y^{2n+1}, ξ) . Then for each formal Legendrian isotopy class there is a loose Legendrian knot in that class, unique up to ambient contact isotopy.*

We will assume that the reader is familiar with the general philosophy of the h-principle, theorems from [18], [14], and [15] are cited explicitly in the paper. A brief outline of the paper follows. In Section 2 we cover a number of definitions from contact topology, including a precise definition of loose knots. We demonstrate an h-principle for ϵ -Legendrian knots in Section 3; this allows us to set up controllably transverse local charts.

Section 4 is a review of [15], defining the concept of wrinkled embeddings and stating an h-principle they satisfy. In the following section we adapt this concept to the Legendrian context, and prove an h-principle for Legendrians with prescribed singularities. Section 6 then describes a method to resolve these singularities. The main theorem is proved in Section 7 using the tools from the previous sections.

In the Conclusion 8, we discuss corollaries of Theorem 1.2, and compare the result to other concepts in contact topology. Finally, we classify formal isotopy classes of Legendrian knots in $(\mathbb{R}^{2n+1}, \xi_{std})$ in the Appendix A.

The author would like to thank her advisor, Yasha Eliashberg, for discussions and support. His influence on this paper is more than manifest.

2. DEFINITIONS FROM CONTACT TOPOLOGY

In this section we give some definitions and general facts about Legendrian knots. By *Darboux neighborhood* in (Y, ξ) we mean an open set $U \subseteq Y$, together with a contactomorphism to a (geometrically) convex subset of $(\mathbb{R}^{2n+1}, \xi_{std})$. Given a Darboux neighborhood we can define two projections, the *Lagrangian projection* $(x_i, y_i, z) \mapsto (x_i, y_i)$ and the *front projection* $(x_i, y_i, z) \mapsto (x_i, z)$. For the former a Legendrian will project to an exact Lagrangian immersion in \mathbb{R}_{std}^{2n} , and the z coordinate can be recovered by integrating $\sum y_i dx_i$. A self intersection in this projection is called a *Reeb chord*.

In the front projection a Legendrian projects to a (highly) singular hypersurface, which nevertheless has well defined tangent fibers everywhere. They are nowhere vertical and the coordinate slopes recover the y_i coordinates of the Legendrian. A Legendrian has a Reeb chord wherever it is self-tangent after a local vertical translation, in particular a Legendrian immersion has a self intersection exactly where its front is self tangent. The kernel of the differential of the front projection is a Legendrian foliation \mathcal{F} whose leaves are the Legendrians $\{(x, z) = \text{constant}\}$. A Legendrian thus has singularities in the front projection exactly where it intersects \mathcal{F} non-transversely. In this paper we only make use of cusp singularities in the

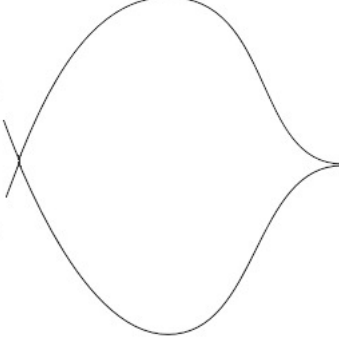


FIGURE 1. A stabilization, in the front projection.

front, locally given by the equation $z^2 = x_1^3$.

Definition 2.1. In \mathbb{R}_{std}^3 , a *stabilization* is a Legendrian curve which has the properties depicted in Figure 1. Specifically, it is required to have a unique transverse self-intersection and a single cusp in the front projection, and a single Reeb chord. The length of this Reeb chord is called the *action* of the stabilization. (We do not distinguish an orientation of a stabilization.)

Remark. Inside a contact 3-manifold (Y, ξ) , a *bypass* is defined to be an embedded topological 2-gon D whose characteristic foliation $TD \cap \xi$ has no singularities on the interior, a negative elliptic singularity on one edge, positive elliptic singularities at the two vertices, and a positive hyperbolic singularity on the remaining edge [20]. See Figure 2. Let L be a Legendrian arc in Y . Then there is a Darboux chart $U \subseteq Y$ so that $L \subseteq U$ is a stabilization if and only if there is a Legendrian arc $\alpha \subseteq Y$ connecting the endpoints of L , so that $L \cup \alpha$ is the boundary of a bypass, where L (respectively α) contains the negative elliptic (positive hyperbolic) singularity. In the coordinate system on U , the arc α is parallel to the y -axis, defining the self-intersection in the front projection. This equivalence is a simple application of the Weinstein neighborhood theorem.

Definition 2.2. Suppose $n > 1$. Let $B \subseteq \mathbb{R}_{std}^3$ be an open ball containing a stabilization of action a , and let $V_\rho = \{|p| \leq \rho, |q| \leq \rho\} \subseteq T^*\mathbb{R}^{n-1}$. Notice $B \times V_\rho$ is an open convex set in \mathbb{R}_{std}^{2n+1} . Let Λ be the cartesian product of the stabilization and the zero section, which is Legendrian in $B \times V_\rho$. We call the pair $(B \times V_\rho, \Lambda)$ a *Legendrian twist*. A Legendrian twist satisfying $\frac{a}{\rho^2} < 2$ is called a *loose chart*. Finally, let L be a Legendrian knot in a contact manifold (Y, ξ) . If there is a Darboux chart $U \subseteq Y$ so that $(U, U \cap L)$ is a loose chart then L is called *loose*.

By changing coordinates and rescaling the contact form, we can make either a or ρ any size, but not simultaneously. The requirement $\frac{a}{\rho^2} < 2$ is the essential condition in the definition; we claim that every Legendrian L contains a subset

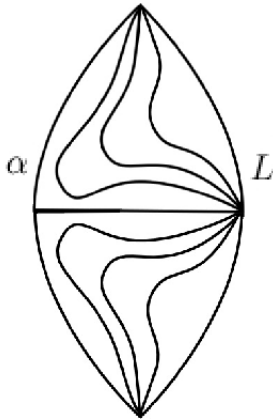


FIGURE 2. The characteristic foliation on a bypass D .

which is a Legendrian twist. To show this it suffices to find a contact 3-ball $B \subseteq Y$ with trivial symplectic normal bundle, so that $B \cap L$ is a stabilization. Let B^3 be any small 3-ball intersecting L in a single arc. Since $n > 1$, the h-principle for isocontact embeddings of positive codimension [18] implies that a C^0 perturbation of B^3 (fixed near L) has the necessary properties.

Proposition 2.3. *Inside a loose chart, there is another Legendrian twist with parameters so that $\frac{a}{\rho^2}$ is arbitrarily small. A loose chart contains two disjointly embedded loose charts.*

Proof: The first statement implies the second. The proof is essentially contained in Figure 3. We can think of ρ being normalized to 1, so that $a < 2$. Then the middle of the Legendrian twist can be squeezed arbitrarily thin on neighborhood with radius bounded below by $\frac{2-a}{4}$ (after smoothing corners). \square

We now define an operation that alters any Legendrian knot so that it becomes loose. This construction is unnecessary for the purpose of constructing a Legendrian isotopy between two loose knots, but we will need it to show the existence portion of Theorem 1.2. This operation was first defined in [11]; there it was introduced (without a name) as an operation to alter Legendrian framings in order to construct Stein manifolds. See Proposition A.3. It was later considered in [8] where it was shown that this operation causes pseudo-holomorphic invariants to become trivial.

In the front, consider a small neighborhood of a cusp singularity. After flattening things out, we can say the neighborhood consists of two horizontal open disks $\{z = 0\}$ and $\{z = 1\}$, connected by a strip with a single cusp. By choosing a smaller neighborhood and rescaling coordinates we can assume this model is arbitrarily large in all x and y directions. Of course any point on a Legendrian admits local coordinates so that the given point is on a cusp in the front, thus a small neighborhood of any point on a Legendrian admits these coordinates.

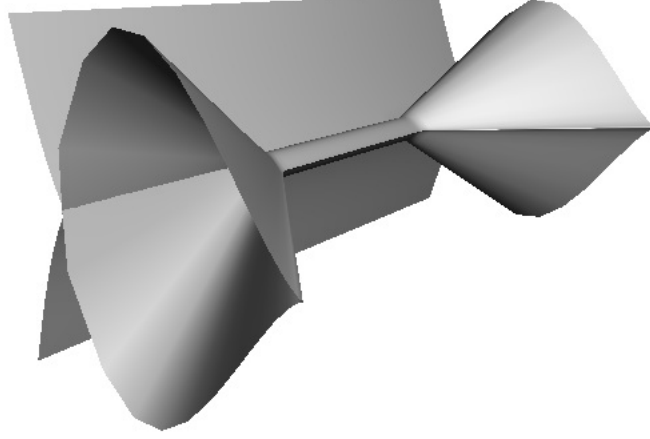


FIGURE 3. We can isotope a loose chart in a neighborhood of itself, so that it contains a Legendrian twist with arbitrarily small action as a subset. This picture is in the front projection, note that all coordinates $y_i = \frac{dz}{dx_i}$ are bounded by ρ

Definition 2.4. Let L^n be a Legendrian knot in (Y, ξ) . Let $M \subseteq D^n$ be a compact, codimension 0 manifold, so that $M \cap \partial D^n = \emptyset$. Choose a Morse function $h : M \rightarrow [0, 2]$, which is identically zero near ∂M and has all critical values larger than 1. Choose a point on L and local coordinates as above, suitably large to accommodate h . On the compactly supported set they disagree, replace the disk $\{z = 0\}$ with the set $\{z = h(x)\}$. This Legendrian knot is called the M -stabilization of L , denoted $s_M(L)$. See Figure 4.

This construction does not depend on the choice of neighborhood, since any small disk in L can be taken to any other by ambient contactomorphism. A priori, $s_M(L)$ may depend on the isotopy class of embeddings $M \subseteq D^n$; we assume this data is included in order to define $s_M(L)$. In fact, Theorem 1.2 implies $s_M(L)$ is determined up to Legendrian isotopy by only $\chi(M)$ and the formal Legendrian isotopy class of L when $n > 1$. For the case $n = 1$ the reader can check that D^1 -stabilizing a knot is equivalent to stabilizing a curve twice, one with each sign.

Proposition 2.5. *For any Legendrian knot of dimension $n > 1$ and any $M \subseteq D^n$, $s_M(L)$ is loose.*

Proof: There is visibly a product neighborhood of stabilizations with action 1, see Figure 4. The radius of the neighborhood in the x directions is determined by the topology of the embedding $M \subseteq D^n$, but the radius in the y directions can be taken to be arbitrarily large from the outset as discussed above. By rescaling the x and y coordinates in inverse proportion (keeping the contact form fixed), we exhibit a loose chart. \square

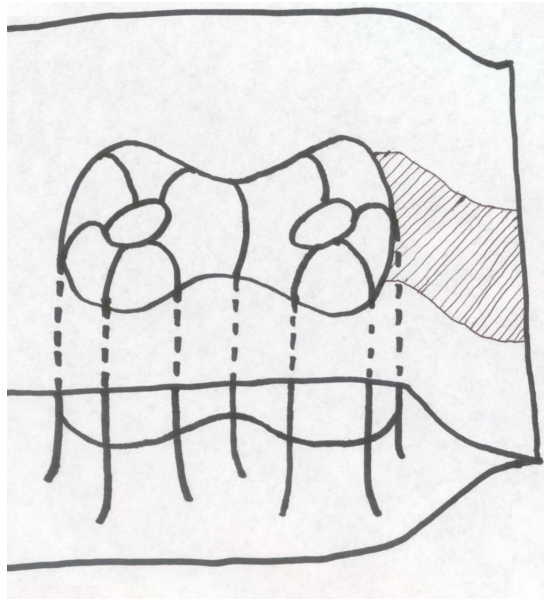


FIGURE 4. An M -stabilization of a small neighborhood. Here M is the thrice punctured sphere. Any M -stabilized Legendrian contains a loose chart, shown here as being underneath the lightly shaded region.

Proposition 2.6. *Let L be a Legendrian knot, and suppose $\chi(M) = 0$. Then $s_M(L)$ is formally isotopic to L .*

Proof: Identify $M \subseteq L$ as the set $\{h(x) > 1\}$, as in Definition 2.4. We first describe a smooth isotopy, undoing the M -stabilization. The y coordinates of our knot are given by the gradient of h . Fixing this near ∂M , we can homotope the gradient to a nonzero vector field, since $\chi(M) = 0$. We interpret this as an isotopy which alters the y coordinates but has a fixed projection. We can then push M down through the $\{z = 1\}$ plane without the knot self-intersecting.

It remains to show that this smooth isotopy, f_t is actually the base of a formal Legendrian isotopy, that is, we need to homotope df_t through bundle monomorphisms to a Lagrangian monomorphism. Since we avoid the singular set, the obvious straight line path through bundle maps projects non-singularly to the x coordinate plane. It follows that this path is in fact through monomorphisms. \square

3. ϵ -LEGENDRIAN KNOTS

We demonstrate an h-principle for ϵ -Legendrian knots in this section. The advantage of working with ϵ -knots rather than formal knots is that it gives us a set of Darboux coordinates around every point, so that L has a smooth front projection. For the purposes of this paper, $\epsilon = \frac{\pi}{3}$ is sufficiently small. First, we define a

Legendrian plane field to be a Lagrangian subfield of the distribution ξ .

Definition 3.1. An embedded submanifold $L^n \subseteq (Y, \xi)$ is called ϵ -*Legendrian* if there is a Legendrian plane field along L , λ , which is ϵ -close to TL . Here, two n -planes are said to be ϵ -close if the projection from one plane to the other is an isomorphism and the angle between any vector and its projection is less than ϵ (in some fixed metric).

We use this opportunity to discuss the general problem of A -directed embeddings, which we will discuss in other contexts throughout the paper. Let L be an n -manifold, and Y a manifold of larger dimension. Let $A \subseteq Gr_n(Y)$, where $Gr_n(Y)$ denotes the *bundle* of n -planes in TY , with fiber $Gr_{\dim(Y),n}$. An *A -directed embedding* is an embedding $L \rightarrow Y$ so that $TL \subseteq A$. A *formal A -directed embedding* is a smooth embedding $f : L \rightarrow Y$, together with a path of bundle monomorphisms $F_s : TL \rightarrow TY$ covering f , so that $F_0 = df$ and $\text{Im}(F_1) \subseteq A$. To say an *h -principle holds* for A -directed embeddings is to say the inclusion of A -directed embeddings into formal A -directed embeddings is a weak homotopy equivalence (with the C^∞ topologies). In particular, it induces a bijection on π_0 of these spaces: for every formal A -directed isotopy class, there is exactly one A -directed embedding up to A -directed isotopy.

Even under the assumption that A is open, an h -principle for A -directed embeddings is not generally true. For example if $L = S^2$ and $Y = \mathbb{R}^3$, the h -principle for A -directed embeddings fails for any proper subset $A \subseteq Gr_{3,2}$. In [18], it is shown that an h -principle holds for all open A , if L is an open manifold. Furthermore, the concept of convex integration is used there to prove an h -principle holds for A -directed embeddings of closed manifolds, under the assumption A is open and *ample*. Rather than stating the original definition, we give the ampleness criterion 19.1.1 from [14].

Proposition 3.2. *Let $A \subseteq Gr_n(Y)$, fix $p \in Y$, and let $S \in Gr_{n-1}(Y)_p$ be a $(n-1)$ -plane contained inside an element of A . Let $\Omega_{p,S} = \{v; \text{Span}\{S, v\} \in A_p\} \subseteq T_p Y$. Assume for every choice of p and S , the convex hull of each connected component of $\Omega_{p,S}$ is equal to $T_p Y$. Then A is ample.*

Let (Y, ξ) be contact, and let $A \subseteq Gr_n(Y)$ be the subset of n -planes which deviate from a Lagrangian plane in ξ by angle less than ϵ . In these terms an embedding $L \rightarrow Y$ is ϵ -Legendrian if and only if it is A -directed. Assume that S is an $(n-1)$ -plane which makes an angle less than ϵ with some Legendrian plane. Then $\Omega_{p,S}$ is connected, open, and scale invariant. This implies the convex hull of $\Omega_{p,S}$ is all of $T_p Y$, and thus A is ample by Proposition 3.2.

Convex integration implies an h -principle for ϵ -Legendrian knots; this means the space of ϵ -Legendrian knots is weakly homotopy equivalent to formal ϵ -Legendrian knots. If furthermore $\epsilon < \frac{\pi}{2}$ then the space of formal ϵ -Legendrian knots is weakly homotopy equivalent to the space of formal Legendrian knots, simply because this

is true in each fiber. We will only use this result on the level of π_0 .

Proposition 3.3. *Let $\epsilon < \frac{\pi}{2}$. Every formal Legendrian is formally homotopic to an ϵ -Legendrian, and any formal isotopy between two ϵ -Legendrians can be C^0 perturbed to an ϵ -Legendrian isotopy, fixed at the endpoints.*

4. REVIEW OF WRINKLED EMBEDDINGS

In this section, we review concepts from [15] needed for the proof of Theorem 1.2. While attempting to be minimally complete, it would be to the reader's advantage to understand the constructions there more thoroughly. Theorem 1.2 can be thought of as an application of Eliashberg/Mishachev's ideas to contact topology.

As discussed in the previous section, an h-principle for A -directed embeddings of a closed manifold L is not generally true, even if we assume A is open. The motivation of the definitions in [15] is to prove an h-principle for all open A , by relaxing the notion of embedding. Specifically, a wrinkled embedding is a smooth map which is a topological embedding, but is allowed to have prescribed singularities. These singularities have well defined tangent fibers, allowing us to define A -directed wrinkled embeddings. The main theorem from [15] is an h-principle for A -directed wrinkled embeddings, for any open A . We now make these statements precise, which we will adapt to a local, codimension 1 situation.

Definition 4.1. Let $W : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be a smooth, proper map, which is a topological embedding. Suppose W is a smooth embedding away from a finite collection of spheres, $\{S_j^{n-1}\}$. Suppose, in some coordinates near these spheres, W can be parametrized by

$$W(u, \vec{v}) = (\vec{v}, u^3 - 3u(1 - |\vec{v}|^2), \frac{1}{5}u^5 - \frac{2}{3}u^3(1 - |\vec{v}|^2) + u(1 - |\vec{v}|^2)^2),$$

where our parameters lie in a small neighborhood of the sphere $\{|\vec{v}|^2 + u^2 = 1\} \subseteq \mathbb{R}^n$. Then W is called a *wrinkled embedding*, and the spheres S_j^{n-1} are called the *wrinkles*.

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}^2$ be the plane curve, defined by $\psi(u) = (u^3 - 3u, \frac{1}{5}u^5 - \frac{2}{3}u^3 + u)$, shown in Figure 5 (we assume ψ is horizontal outside a compact subset). Let ψ_δ be a rescaling of this function, defined by $\psi_\delta(u) = (\delta^{3/2}\psi_1(\frac{u}{\sqrt{\delta}}), \delta^{5/2}\psi_2(\frac{u}{\sqrt{\delta}}))$. This is well defined even when $\delta < 0$, in this case ψ_δ is smooth and graphical. We define $\psi_0(u) = (u^3, \frac{1}{5}u^5)$, which makes ψ_δ a continuous family of plane curves.

In these terms, wrinkled embeddings are locally modeled by $W(u, \vec{v}) = (\vec{v}, \psi_{1-|\vec{v}|^2}(u))$. Therefore wrinkles have two kinds of singularities: on the singular sphere $\{|\vec{v}|^2 + u^2 = 1\}$, there are cusp singularities everywhere on the lower and upper hemisphere. Along the equator $\{u = 0\}$, we see “unfurled swallowtail” singularities. See Figure 6.

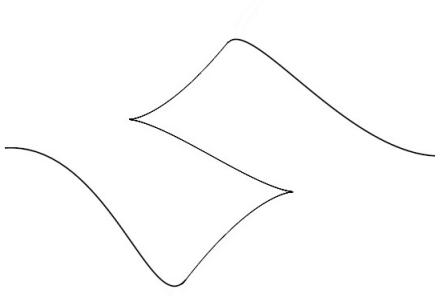
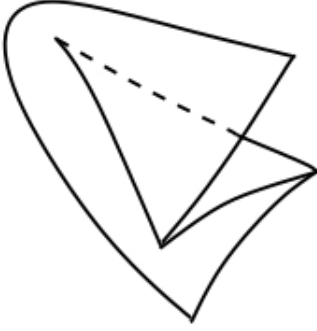

 FIGURE 5. The curve ψ .


FIGURE 6. An unfurled swallowtail singularity.

An *embryo* of a wrinkle is defined to be the isolated singularity with a local model given by

$$(u, \vec{v}) \mapsto (u^3 + 3u|\vec{v}|^2, \vec{v}, \frac{1}{5}u^5 + \frac{2}{3}u^3|\vec{v}|^2 + u|\vec{v}|^4)$$

with (u, \vec{v}) in a neighborhood of the origin. For $t \in (-\epsilon, \epsilon)$, let $W_t(\vec{v}, u) = (\vec{v}, \psi_{t-|\vec{v}|^2}(u))$. Then W_t is smooth for $t < 0$ and has a single wrinkle when $t > 0$. At $t = 0$, there is an embryo singularity at $(u, \vec{v}) = 0$. We allow embryo singularities whenever we discuss parametric families of wrinkled embeddings. Generically, these occur with codimension 1 in parameter space, and are isolated points in the embedding. We do not distinguish a time orientation, so an embryo can either create a wrinkle in forward time, or allow one to disappear.

Even though a wrinkled embedding is singular, it does have well defined tangent fibers of dimension n everywhere. For example, let p be a cusp singularity point given in coordinates by $f(u) = (u^2, u^3)$. Even though df is trivial at the point $u = 0$, small neighborhoods of this point are C^1 close to uniformly horizontal. Therefore we define the tangent fiber to be horizontal at that point. One can similarly check that the tangent fibers near an unfurled swallowtail or embryo singularity uniformly

approach horizontal in the coordinates given above.

Thus given a wrinkled embedding $W : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$, we can define the hyperplane bundle $TW \subseteq T\mathbb{R}^{n+1}|_{\text{Im}(W)}$. This allows us to define *A-directed wrinkled embeddings*: wrinkled embeddings with $TW \subseteq A$. We quote the h-principle from [15], again specialized for our purposes:

Theorem 4.2. *Let $A \subseteq Gr_n(\mathbb{R}^{n+1})$ be any open subset of the hyperplane Grassmanian bundle which is fiberwise non-empty, and let $C \subseteq \mathbb{R}^n$ be a compact set. If (f, F_s) be a formal A-directed embedding so that $F_s = df$ outside C , then there is an A-directed wrinkled embedding $W : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ so that f and W are C^0 close and identical outside C . If $(f_t, F_{s,t})$ is a formal A-directed isotopy between the A-directed embeddings f_0 and f_1 which is supported on C , then there is a C^0 perturbation of f_t to an A-directed wrinkled isotopy W_t , which equals f_t outside C and near $t = 0, 1$.*

5. WRINKLED LEGENDRIANS

Say we wanted to prove that embedded Legendrian knots satisfy an h-principle, despite knowing this is false. We will see this reduces simply to solving the local extension problem: given a formal Legendrian $f : \mathbb{R}^n \rightarrow \mathbb{R}_{std}^{2n+1}$ which is Legendrian outside a compact set, we need to show we can find a Legendrian embedding C^0 close to f , and equal to it outside the compact set. The C^0 close condition is essential: we have no lower bounds on the size of our local charts and we need to avoid self intersections.

The set of Legendrian planes in $Gr_n(\mathbb{R}_{std}^{2n+1})$ is not open, so none of our theorems about directed embeddings apply immediately. The advantage of the local picture is it allows us to re-interpret the geometry in the front projection. Any smooth embedding $W : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ which is never vertical defines a Legendrian $L \subseteq \mathbb{R}_{std}^{2n+1}$. Assume f projects to a smooth hypersurface $H \subseteq \mathbb{R}^{n+1}$. The y coordinates of f define a hyperplane field $\nu = \ker(dz - \sum_i y_i dx_i) \subseteq T\mathbb{R}^{n+1}|_H$. Then L is C^0 close to f if W is C^0 close to H and TW is C^0 close to ν . If we let $A = \text{"}n\text{-planes } C^0 \text{ close to } \nu\text{"}$, W is equivalently an A -directed embedding C^0 close to H .

In fact W need not be smooth, since a smooth Legendrian need not have a smooth front projection. At this point we would like to use Theorem 4.2, but first we need to study wrinkled singularities to determine if they have smooth Legendrian lifts. Wrinkled embeddings have a natural tangent bundle. More precisely, given a non-vertical wrinkled embedding $W : \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ there are unique smooth functions $y_i(\vec{v}, u)$ so that $\frac{\partial z}{\partial x_i} = y_i$. At a cusp singularity $(x, z) = (u^2, u^3)$, this function is given by $y = \frac{3}{2}u$. In this case, the triple (x, y, z) is a smooth embedding, thus the cusp singularity is the front projection of a smooth Legendrian curve.

For unfurled swallowtails (as well as embryos) the functions y_i are uniquely defined, but the induced map $L : \mathbb{R}^n \rightarrow \mathbb{R}_{std}^{2n+1}$ is not an embedding: dL has rank $n - 1$ at these singularities. For now we “define away” this problem.

Definition 5.1. Let L be closed and connected. A *wrinkled Legendrian* is a smooth map $f : L \rightarrow (Y, \xi)$, which is a topological embedding, satisfying the following properties. The image of df is contained in ξ everywhere, and df is full rank outside a subset of codimension 2. This singular set is required to be diffeomorphic to a disjoint union of $(n-2)$ -spheres, called *Legendrian wrinkles*. Near the singular set we require f is modeled by a map $(u, x_2, \dots, x_n) \mapsto (x_1, y_1, \dots, x_n, y_n, z)$ defined by

$$\begin{aligned} x_1 &= u^3 - 3ux_2 \\ y_1 &= \frac{1}{3}(u^2 - x_2) \\ y_2 &= ux_2 + \frac{1}{3}u^3 \\ y_i &= 0 \text{ for } i > 2 \\ z &= \frac{1}{5}u^5 - \frac{2}{3}u^3x_2 + ux_2^2 \end{aligned}$$

for some Darboux coordinates on Y , with $u \in (-\epsilon, \epsilon)$.

Any wrinkled embedding in \mathbb{R}^{n+1} has a unique lift to a wrinkled Legendrian in \mathbb{R}_{std}^{2n+1} . With that in mind, we define a *Legendrian embryo* to be the singularity given by the model

$$\begin{aligned} x_1 &= u^3 + 3u|\vec{v}|^2 \\ (x_2, \dots, x_n) &= \vec{v} \\ y_1 &= \frac{1}{3}(u^2 + |\vec{v}|^2) \\ (y_2, \dots, y_n) &= 2u(|\vec{v}|^2 - \frac{1}{3}u^2)\vec{v} \\ z &= \frac{1}{5}u^5 + \frac{2}{3}u^3|\vec{v}|^2 + u|\vec{v}|^4 \end{aligned}$$

which is the Legendrian lift of an embryo of a wrinkled embedding (coordinates (u, \vec{v}) in a neighborhood of the origin in \mathbb{R}^n). As before, we allow Legendrian embryos whenever we discuss parametric families of wrinkled Legendrians, so that Legendrian wrinkles may appear and disappear during a wrinkled isotopy.

We claim that Legendrian wrinkles have a canonical coorientation in L . Using the given coordinates in Definition 5.1 we see $\{u = 0, x_2 = 0\}$ is the singular set. df has rank $n-1$ on this set, and its kernel is spanned by ∂_u . Let $\beta : (-\epsilon, \epsilon) \rightarrow L$ be a path with $\beta(0) = 0$, so that $\dot{\beta}$ is in the kernel of df at that point. Then the second derivative of $f \circ \beta$ at $s = 0$ defines a nonzero vector $v \in TY$. For another choice of path β_1 , v is scaled by $\left(\frac{|\dot{\beta}_1(0)|}{|\dot{\beta}(0)|}\right)^2$ and added to $df(\ddot{\beta}_1(0))$. Thus at the wrinkle, there is a canonical n -plane containing the $\text{Im}(df)$, and it is canonically cooriented.

Also notice that $\text{Im}(df)$ contains the tangent space of the singular set, and thus a Legendrian wrinkle has a canonical normal framing in L .

For the purposes of our proof, Legendrian wrinkles will only be created in local neighborhoods and thus the Legendrian wrinkles we use will satisfy topological restrictions. The next definition is intended to keep track of this data, as it will be

necessary for our constructions later.

Definition 5.2. Let L be a wrinkled Legendrian knot, with wrinkles $\{S_j^{n-2}\}$. A *disk framing* is a disjoint collection of disks $\{D_j^{n-1}\}$, so that $\partial D_j^{n-1} = S_j^{n-2}$, and this induces the same framing and orientation as $\text{Im}(df)$. For families L_t , we require $D_{j,t}^{n-1}$ contracts to a point whenever $S_{j,t}^{n-2}$ contracts to an embryo.

Theorem 5.3. *Suppose L_0, L_1 are formally isotopic Legendrian knots in (Y, ξ) . Then there is a wrinkled Legendrian isotopy between them with a compatible disk framing.*

This theorem is essentially a combination of the h-principles discussed so far. We first prove a lemma to reduce the problem to Darboux charts. As explained in Section 3, ϵ should be thought of as being roughly $\frac{\pi}{3}$; we reserve the words “close” and “small” for an arbitrarily small size, which may depend on ϵ . We say two planes are ϵ -orthogonal if every respective pair of vectors define an angle greater than $\frac{\pi}{2} - \epsilon$.

Lemma 5.4. *Let $L_t : L \rightarrow (Y, \xi)$ be an isotopy of ϵ -Legendrian knots, Legendrian at the endpoints. Then, we can perturb L_t through ϵ -Legendrians (C^0 small, fixed at the endpoints) to \tilde{L}_t , and find a collection of moving disjoint Darboux balls, B_k^t , so that \tilde{L}_t is Legendrian outside $\bigcup_k B_k^t$. Furthermore, we can assume that $\tilde{L}_t \cap B_k^t$ always has a graphical front projection.*

Proof: Let $\theta_t : Y \rightarrow Y$ be a smooth ambient isotopy extending L_t . For each $p \in L_0$, $t \in [0, 1]$, let $U_{p,t} : B_{std}^{2n+1} \rightarrow Y$ be a small Darboux neighborhood around $\theta_t(p)$, so that $U_{p,t}(0) = p$ and $(U_{p,t})_*(\text{span}_i\{\partial_{x_i}\}) = T_p L_t$ at this point. Choose $U_{p,t}$ small enough so that λ_t and $(U_{p,t})_* \text{span}_i\{\partial_{x_i} + y_i \partial_z\}$ are ϵ -close, and also $\partial_z \notin \lambda_t$. The set $\{\bigcap_t \theta_t^{-1}(U_{p,t}(B_{std}^{2n+1})), p \in L_0\}$ is an open cover of L_0 , choose a finite subcover, indexed by the points p_k . The cover defines a triangulation of L_0 , so that each $(n-i)$ -simplex is completely contained in $i+1$ of these charts, let K be the codimension 1 skeleton. Note that for all t , $\{U_{p_k,t}\}$ is a finite covering of L_t by Darboux balls.

Fix t . In coordinates, $L_t \cap U_k = \{z = z(x), y_i = y_i(x)\}$, and $\lambda = \text{span}_i\{\partial_{x_i} + y_i(x)\partial_z + \sum_j g_i^j \partial_{y_j}\}$, for some functions $g_i^j(x)$. The Lagrangian condition on λ implies $g_i^j = g_j^i$, therefore the functions $z(x), y_i(x), g_i^j(x)$ together define a section of the second jet space of the x coordinates, $J^2(B^n)$.

To construct \tilde{L}_t , we perturb L_t inductively on each Darboux chart. Let $L_t^k = U_{p_k,t}^{-1}(L_t) \subseteq B_{std}^{2n+1}$. Then the ϵ -isotopy (L_t^k, λ_t) , defines a path of non-holonomic sections of $J^2(B^n)$. We cite the 1-parametric Holonomic Approximation Theorem (Theorem 3.1.2 in [14]). It states that, on some open neighborhood V of some C^0 perturbation of $K \cap L_t^k$, we may C^0 approximate the path of sections $(z, y_i, g_i^j)_t$ with sections $(\tilde{z}, \tilde{y}_i, \tilde{g}_i^j)_t$ which are holonomic on V . Over the intersection with previous charts, we fix our section where it is already holonomic and use the extension form

of Holonomic Approximation. This defines a ϵ -Legendrian isotopy \tilde{L}_t which is C^0 close to L_t . \tilde{L}_t is Legendrian on V , since the section defining it is holonomic there. Furthermore $T\tilde{L}_t$ is C^0 close to λ_t over V , thus \tilde{L}_t is graphical on V , in every chart $U_{p_k,t}$. To find B_k^t , take an open subset of $U_{p_k,t}$ which does not intersect K , but whose boundary is contained in V . \square

Proof of Theorem 5.3: Suppose L_0, L_1 are formally isotopic. First, we cite Proposition 3.3 to find an ϵ -isotopy L_t between them. The previous lemma then constructs a moving collection of Darboux balls, so that L_t is Legendrian outside. Thus in each ball we have an ϵ -Legendrian isotopy, which is Legendrian outside of a compact region. The front projection of L_t is a smooth graphical isotopy of a hypersurface, the y -coordinates determine a nowhere vertical hyperplane field along this front. We then apply Theorem 4.2 to find an isotopy W_t of wrinkled hypersurfaces, so that W_t is C^0 close to the front of L_t , and TW_t is C^0 close to the given hyperplane field. This ensures the wrinkled Legendrian lift of W_t is C^0 close to L_t , and therefore remains embedded.

Each Legendrian wrinkle is born and dies in a single Darboux chart. By Definition 4.1 the singular set in the front projection is made up $(n-1)$ -spheres, so that the unfurled swallowtail set (the projection of the Legendrian wrinkle) is the equator. Either hemisphere is a component of the cusp singular set, choosing one hemisphere for each wrinkle defines a disk framing. \square

This theorem holds in all dimensions. In the case $n=1$, a *generic* wrinkled Legendrian will be smooth, since Legendrian wrinkles are codimension 2 submanifolds. However, an isotopy will contain Legendrian embryos, so we cannot conclude L_0 is Legendrian isotopic to L_1 . In fact, crossing through a Legendrian embryo in this dimension is equivalent to a Legendrian curve stabilization (or destabilization). This implies a theorem originally proved in [17]: if two Legendrian 1-knots are formally isotopic, they are Legendrian isotopic after a finite number of stabilizations (with both orientations). It's also true that two smoothly isotopic Legendrian 1-knots are formally isotopic after some number of stabilizations, this is a simple calculation in the algebraic topology of frame bundles (see Appendix A).

6. TWIST MARKINGS

Theorem 5.3 is a major portion of the work in proving Theorem 1.2. In this section, we systematize a method to resolve the singularities of wrinkled Legendrians.

Definition 6.1. Let $L \subseteq (Y, \xi)$ be a wrinkled Legendrian with k wrinkles, and let $\Phi \subseteq L$ be an embedded codimension 1 smooth compact submanifold with boundary. Assume Φ has the topology of a sphere with k open disks removed. Then Φ is called a *twist marking* if the singular set of L is equal to $\partial\Phi$, and there is a small collar neighborhood of the singular set so that $\Phi = \{u=0, x_2 \leq 0\} \subseteq L$ in terms of the coordinates from Definition 5.1. If L has a Legendrian embryo, we require that it is contained in the interior of Φ with local model given by $\Phi = \{u=0\}$.



FIGURE 7. A local model, describing how a twist marking resolves Legendrian wrinkles.

Definition 6.2. We use the following topology on the space of wrinkled Legendrians with twist markings. We put the C^∞ topology on both the space of wrinkled Legendrians as well as the space of embedded submanifolds. We also specify a relation to accomodate discrete changes in the topology of Φ : if L_t is a path of wrinkled Legendrians containing a Legendrian embryo, and Φ_t is a path of twist markings on L_t so that Φ_t acquires another puncture at the embryo, the path (L_t, Φ_t) is defined to be continuous.

Let us take the time to properly compare a twist marking and a disk framing (Definition 5.2) on a wrinkled Legendrian knot, L . Both are codimension 1 submanifolds of L with boundary equal to the singular set of L . Furthermore both are required to meet the same framing condition at this set. The main difference is their topology: a disk framing is a disjoint union of disks, whereas a twist marking is a connected submanifold. Over a small isotopy with a Legendrian embryo a disk framing will lose one component; a twist framing will have a boundary component close up and become a smooth interior point.

Recall Legendrian wrinkles have neighborhoods with front projection given by $\{(x_1, \dots, x_n, z); (x_1, z) = \psi_{x_2}(u), u \in (-\epsilon, \epsilon)\}$, where $\{(x_2, u) = (0, 0)\}$ is the singular set. Let $\delta > 0$ be some small number. Φ should be thought of as a formal representation of the neighborhood $\{(x_1, z) = \psi_\delta(u); u \in (-\epsilon, \epsilon)\}$, where the x_1 direction is transverse to Φ . This interpretation allows us to resolve all singularities at $\partial\Phi$.

Proposition 6.3. *In (Y, ξ) , let (L_t, Φ_t) be an isotopy of wrinkled Legendrians with twist markings. Then we can construct an isotopy of Legendrian knots $\tilde{L}_t : L \rightarrow (Y, \xi)$, so that \tilde{L}_t identical to L_t outside any small neighborhood of Φ_t .*

Proof: We only need to check things for our given models; it suffices to work in the front projection. Note that two (possibly wrinkled) Legendrian embeddings are C^0 close if their fronts are C^1 close. Check that when $\delta > 0$ is small, $\psi_\delta(u)$ is C^1 close

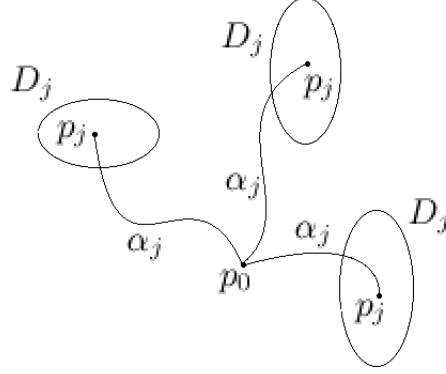


FIGURE 8. Building a Legendrian twist inside L . Pictured here is the case $n = 3$.

to the horizontal axis, and identical to it outside a small neighborhood of the origin. On the interior of Φ , we find coordinates so that $(L, \Phi) = (\{z = 0\}, \{(x_1, z) = (0, 0)\})$, and replace this by $\tilde{L} = \{(x_1, z) = \psi_\delta(u); u \in \mathbb{R}\}$. This is a C^1 small alteration contained in a neighborhood of Φ ; it remains to describe the behavior near $\partial\Phi$.

Let $m_\delta : \mathbb{R} \rightarrow \mathbb{R}$ be a smoothing of the function $\max(\delta, \cdot)$. Near points on $\partial\Phi$, we have coordinates so that $(L, \Phi) = (\{(x_1, z) = \psi_{x_2}(u)\}, \{u = 0, x_2 \leq 0\})$, which we replace with $\tilde{L} = \{(x_1, z) = \psi_{m_\delta(x_2)}(u)\}$. This is nonsingular and compatible with our definition of the interior of Φ . Since both L and \tilde{L} are C^1 close to horizontal near the singular set this alteration is C^1 small. See Figure 7.

Finally, we check this construction near a Legendrian embryo singularity. Let L_t be a path with a unique embryo, we may choose coordinates so that the front of L_t is given by $\{(x_1, z) = \psi_{t-r^2}(u); r^2 = x_2^2 + \dots + x_n^2, u \in \mathbb{R}\}$, and $\Phi_t = \{x_1 = 0, x_2^2 + \dots + x_n^2 \geq t\}$. We then replace this path of wrinkled Legendrians with the path \tilde{L}_t , with front $\{(x_1, z) = \psi_{m_\delta(t-r^2)}(u)\}$. \square

The next proposition is a topological observation without much depth. However, it is the only place the assumption $n > 1$ is essentially used. It is also the only step in the proof that does not obviously extend to $t \in D^m$ for $m > 1$.

Proposition 6.4. *Let $n > 1$. Suppose $L_t : L \rightarrow (Y, \xi)$ is an isotopy of wrinkled Legendrian knots equipped with a disk framing. Then there is a twist marking $\Phi_t \subseteq L_t$, so that Φ_t together with the disk framing represents the class $0 \in \pi_{n-1}L$.*

Proof: For each t , let $\{D_j^{n-1}\}$ be the disk framing of L_t . These define the same framing of S_j^{n-2} as $\text{Im}(dL_t)$, thus we can C^0 perturb this disk near its boundary

so that they approach the Legendrian wrinkle as specified in Definition 6.1.

For all j pick points $p_j^t \in D_j^{n-1}$, also choose a path of basepoints $p_0^t \in L_t$ disjoint from these disks. Each wrinkle S_j^{n-2} is canonically cooriented; this induces a coorientation on D_j^{n-1} . We first describe Φ_0 . For each j , we find a curve α_j , connecting p_0^0 to p_j^0 . We require that the α_j are mutually disjoint, and do not intersect any D_j^{n-1} on their interior. Furthermore we ask that α_j is transverse to D_j^{n-1} at p_j^0 , and the outward tangent to α_j matches the coorientation on D_j^{n-1} . Let S be the boundary of a small neighborhood of $\alpha = \bigcup_j \alpha_j$ (if L_0 is smooth then $\alpha = p_0^0$). For any j , $S \cap D_j^{n-1}$ is a small $(n-2)$ -sphere which bounds a small disk in both S and D_j^{n-1} . Discard these disks, and smooth corners to get a connected smooth manifold. After doing this for all j we obtain a manifold Φ_0 , satisfying all the conditions in Definition 6.1.

We now construct Φ_t . An isotopy of wrinkled embeddings has embryo singularities at points isolated in both space and time. On any subinterval of time not containing an embryo, the isotopy is induced by an ambient isotopy of \mathbb{R}^{n+1} . On such intervals we can simply let the ambient isotopy act on α , which naturally gives us an isotopy of twist markings. Therefore, it suffices to describe Φ_t in a small time interval around an embryo singularity at time t_0 .

We first consider an embryo singularity where a wrinkle S_j^{n-2} disappears in forward time. At the embryo $T\Phi_{t_0} = \text{Im}(dL_{t_0})$, since this equation is satisfied for all points on $\partial\Phi_t$ when $t < t_0$. For $t > t_0$, Φ_t has a long “tentacle” which isn’t attached to anything. This can be retracted inside a neighborhood of p_0^t , and the isotopy can be continued. For wrinkle creation note that this process can be reversed. Immediately before an embryo occurs we can extend a tentacle out from p_0 to contain it. Furthermore while keeping everything embedded, we can do this so that $T\Phi_{t_0}$ is tangent to $\text{Im}(dL_{t_0})$, with given orientation. \square

We now prove a proposition that relates resolutions of Legendrian wrinkle singularities with loose charts. We say two properly embedded Legendrians (U_0, L_0) and (U_1, L_1) in the Darboux charts U_i are *equivalent* if we can find contact inclusions $\iota_0 : U_0 \rightarrow U_1$ and $\iota_1 : U_1 \rightarrow U_0$, so that $L_0 = \iota_0^{-1}(L_1)$ and $L_1 = \iota_1^{-1}(L_0)$. As our main example, notice the Legendrian in \mathbb{R}_{std}^3 with front projection $\{(x, z) = \psi(\mathbb{R})\}$ is equivalent to any stabilization (Definition 2.1).

Consider the properly embedded wrinkled Legendrian $\Lambda : B^n \rightarrow B_{std}^{2n+1}$ with front given by $\{(x_1, z) = \psi_{r^2-1}(u); r^2 = x_2^2 + \dots + x_n^2, u \in \mathbb{R}\}$; we refer to this as an *inside-out wrinkle*. Recall that Legendrian wrinkles are given by $\{(x_1, z) = \psi_{1-r^2}(u)\}$ if we assume they have a disk framing. An inside-out wrinkle contains a single Legendrian wrinkle at $\{|\vec{v}|^2 = 1, u = 0\}$, however the embedding is not compactly supported. Notice the front projection is singular on the set $\{|\vec{v}|^2 - u^2 = 1\}$. Let $D_0^{n-1} = \{|\vec{v}| \leq 1, u = 0\}$, which is easily checked to be a twist marking on Λ . We use (Λ, D_0^{n-1}) as a model for a twist marking single boundary component. We now show it is equivalent to a loose chart.

Proposition 6.5. *Let (Λ, D_0^{n-1}) be an inside out wrinkle with the twist marking defined above. Let $\tilde{\Lambda}$ be the resolution of D_0^{n-1} at any scale, and let B be any Darboux ball containing D_0^{n-1} . Let (U, L) be any loose chart. Then $(B, \tilde{\Lambda})$ and (U, L) are equivalent models in terms of mutual pairwise inclusion by contactomorphism.*

Proof: $\tilde{\Lambda}$ is a product neighborhood of the curve ψ , whereas a loose chart is a product neighborhood of a stabilization. By taking a smaller subset of either model and rescaling, we can assume both product neighborhoods are arbitrarily wide. This is the content of Proposition 2.3 for the case of a loose chart. The fact that $\tilde{\Lambda}$ is a resolution of a twist marking is essentially an exhibit that it can be isotoped inside B to an arbitrarily small height compared to the neighborhood width. \square

7. COMPLETING THE MAIN PROOF

We now complete the proof of Theorem 1.2. We start by proving uniqueness of loose knots in a given formal isotopy class.

Theorem 7.1. *Let $n > 1$. Suppose $L_0, L_1 \subseteq (Y, \xi)$ are two formally isotopic loose Legendrian knots. Then they are Legendrian isotopic.*

Proof: Suppose L_0, L_1 are loose Legendrian knots with a formal isotopy between them, and let B be a Darboux ball so that $B \cap L_0$ is a loose chart. After composing with an ambient contact isotopy of Y , we can assume that $B \cap L_1 = B \cap L_0$. We then can pick an ambient smooth isotopy ζ_t^s equal to the identity on $\{t = 0, 1\} \cup \{s = 0\}$, so that $B \cap \zeta_t^1(L_t) = B \cap L_0$. Though ζ_t^s cannot be made into a contact isotopy, $\zeta_t^1(L_t)$ is a formal Legendrian isotopy with bundle homtoopy $d\zeta_{1-s}^t \circ F_s^t$, where $F_s^t : TL \rightarrow TY$ is the bundle homtoopy for L_t . Then $\zeta_t^1(L_t)$ is a formal isotopy between L_0 and L_1 , we relabel it L_t .

Now we apply Theorem 5.3 to find a wrinkled Legendrian isotopy between L_0 and L_1 , which we continue to denote L_t . By applying the theorem as an extension, L_t retains the property that $B \cap L_t$ is a fixed loose chart. By Proposition 6.5 we can find a smaller ball $\tilde{B} \subseteq B$ so that $\tilde{B} \cap L_t$ is isotopic to the standard resolution of an inside-out wrinkle. Let Λ_t be the wrinkled isotopy which is equal to L_t outside \tilde{B} , and $\tilde{B} \cap \Lambda_t$ is that inside-out wrinkle, unresolved. The disk framing on L_t extends to one on Λ_t , by adding one additional disk inside \tilde{B} . We then apply Proposition 6.4 to get a path of twist markings $\Phi_t \subseteq \Lambda_t$. Since Λ_0 is smooth outside \tilde{B} , Φ_0 is a disk with boundary inside \tilde{B} representing $0 \in \pi_{n-1}(L, B)$. Thus after isotopy we can assume $\Phi_0 \subseteq B$, similarly for Φ_1 . Resolve the wrinkled isotopy Λ_t along Φ_t using Proposition 6.3, this defines a Legendrian isotopy $\tilde{\Lambda}_t$. By Proposition 6.5, $\tilde{\Lambda}_0 = L_0$, similarly $\tilde{\Lambda}_1 = L_1$. \square

Proposition 7.2. *Let $n > 1$, and suppose (f, F_s) is a formal Legendrian knot in (Y, ξ) . Then there is a Legendrian knot which is formally isotopic to (f, F_s) .*

For any Legendrian L , $s_{S^1 \times D^{n-1}}(L)$ is a loose knot in the same formal isotopy class as L , so this proposition implies the existence theorem for loose Legendrian knots. This is essentially proved in [11], let us first outline their proof here. Since immersed Legendrians satisfy an h-principle, we focus on the set of formal Legendrian isotopy classes in a fixed regular Legendrian homotopy class. A simple calculation shows this set of formal isotopy classes admits a transitive \mathbb{Z} action, and we then show that M -stabilization generates this action by $\chi(M)$. Thus given any formal Legendrian isotopy class we can first find a Legendrian immersion L in the correct regular homotopy class, which will generically be embedded. Then we just pick an M so that $s_M(L)$ is in the correct formal isotopy class; notice we can realize any integer by $\chi(M)$ since $n > 1$. Much of this proof is explained in the Appendix A, though there are a number of gaps. To the author's knowledge a complete proof does not exist in the literature, though it's been a "known theorem" since [11].

Here we give a different proof of the statement using our h-principle method. It is a distinct proof in that it does not require any knowledge about the set of all formal isotopy classes. In particular, the case for general Y and L is no more difficult than the case of spheres in \mathbb{R}_{std}^{2n+1} ; the above proof is more difficult to extend to cases where $\pi_1 Y \neq 0$ or when L is not nulhomologous.

Proof: We would like to mirror the proof of Theorem 7.1. f is formally isotopic to an ϵ -Legendrian, by Proposition 3.3. The existence analogue of Lemma 5.4 has the same proof, therefore we can assume f is Legendrian outside of some disjoint Darboux balls, and transverse to \mathcal{F} inside those balls. The difficulty is an analogue of Theorem 5.3 does not make sense with our current definitions: we cannot say that f is formally isotopic to a wrinkled Legendrian, because there is no obvious interpretation of wrinkled Legendrians as formal Legendrians. We avoid the issue by instead only using these concepts locally.

Lemma 7.3. *Let $n > 1$. Suppose L_0 is an ϵ -Legendrian embedding $\mathbb{R}^n \rightarrow \mathbb{R}_{std}^{2n+1}$, which is graphical with respect to x , and Legendrian outside of a compact subset, C . Then there exists a C^0 small formal isotopy supported on C from L_0 to a Legendrian embedding.*

Proof: First, we construct a formal isotopy L_t , so that L_1 is a Legendrian embedding C^0 close to L_0 , we adjust the isotopy to be C^0 small at the end of the proof. By taking a smaller ϵ if necessary and reapplying Proposition 3.3, we can assume λ is transverse to \mathcal{F} . Project L_0 to the front, where it is smooth and graphical. Let $L_{1/4}$ be the canonical Legendrian lift in \mathbb{R}_{std}^{2n+1} of this projection (defined by $y_i = \frac{\partial z}{\partial x_i}$). We claim that L_0 is formally isotopic to $L_{1/4}$. Let f_t be the graphical smooth isotopy between them, notice df_t is transverse to \mathcal{F} everywhere. Consider the space of all Lagrangian planes in a symplectic vector space, transverse to a fixed Lagrangian plane. This can be identified as the set of skew-symmetric matrices on \mathbb{R}^n , a contractible space. df_t and λ are thus sections into a bundle with contractible fiber, therefore f can be made into a formal isotopy.

Choose a small neighborhood U of a point in ∂C , so that $L_{1/4}$ is C^0 close to L_0 on U . Perform an $(S^1 \times D^{n-1})$ -stabilization in $U \cap C$, small enough to retain the

C^0 closeness to L_0 . We call this new Legendrian $L_{1/2}$. By Proposition 2.6, $L_{1/4}$ is formally isotopic to $L_{1/2}$.

Next, consider the front projection of $L_{1/2}$, Λ_0 . Outside of U , Λ_0 has the same front as L_0 , thus it is smooth there. The y coordinates of L_0 define a hyperplane field along Λ_0 , we apply Theorem 4.2 to construct a wrinkled isotopy Λ_t , supported on $C \setminus U$, so that $T\Lambda_1$ is C^0 close to the specified hyperplane field. We lift Λ_t to a wrinkled Legendrian isotopy (also denoted Λ_t), so that Λ_1 is C^0 close to L_0 , $\Lambda_0 = L_{1/2}$ outside U , and so that a subset of $\Lambda_t \cap U$ has been replaced by an inside-out wrinkle, as in Proposition 6.5.

We apply Proposition 6.4 to Λ_t , and use the resulting twist marking to resolve the wrinkles, as in Proposition 6.3. We call the resulting Legendrian isotopy L_t , $t \in [\frac{3}{4}, 1]$. L_1 is a Legendrian C^0 close to L_0 , and $L_{3/4}$ is Legendrian isotopic to $L_{1/2}$ by Proposition 6.5.

Finally we construct a C^0 small isotopy; suppose we want the isotopy to be within δ of L_0 . Let V_r be the r neighborhood of L_0 . We construct a formal Legendrian isotopy as above, so that $L_1 \subseteq V_{\delta/2}$. Choose R so that V_R contains the entire isotopy L_t . Choose a smooth isotopy $\theta_t : V_R \rightarrow V_R$, so that θ_0 is the identity, θ_t is fixed to be the identity on $V_{\delta/2}$, and $\theta_1(V_R) \subseteq V_\delta$. Then $\theta_1(L_t)$ is the desired formal isotopy, with bundle homotopy given by $(d\theta_{1-s}) \circ F_s^t$, where F_s^t is the bundle homotopy associated to L_t . \square

This completes the proof of the Proposition 7.2, and thus Theorem 1.2. \square

8. CONCLUSION

We take some time to discuss how Theorem 1.2 relates to other results in the field. The term “loose Legendrian knot” comes from 3-dimensional contact topology [10], there it means a Legendrian knot whose complement is overtwisted. Our concept of loose knots is significantly different: loose knots exist in high dimensional Darboux charts. In both cases looseness is a hypothesis about the global contact topology of the knot complement which allows us to apply h-principle results to Legendrians. Unlike the overtwisted complement case, the geometric model defined here is not contained in any compact region of the complement; instead it is required to intersect the (standard) tubular neighborhood of the knot in a prescribed way.

Presently, overtwistedness in high dimensions is not understood, if such a concept exists at all. To justify our use of terminology, we argue that generally there can only be one flexible class of Legendrian knots. Let (Y, ξ) be a high dimensional contact manifold which is “overtwisted”. Let L be a loose Legendrian knot, and suppose L_{OT} is in the same formal isotopy class, and has overtwisted complement. Since an M -stabilization takes place in a small neighborhood of a point, $L' = s_{S^1 \times D^{n-1}}(L_{OT})$ is a loose knot, whose complement remains overtwisted. By

Theorem 1.2, L is Legendrian isotopic to L' . If knots with overtwisted complement have flexibility properties (as in [12] or [6] in the $n = 1$ case), L_{OT} should be ambient isotopic (or at least contactomorphic) to L' . Thus in a hypothetical high dimensional overtwisted manifold, we expect a Legendrian knot is loose exactly when it has overtwisted complement.

We can extend our result to a h-principle for loose Legendrian links, where the definition requires each component to have a loose chart disjoint from the other components. Note that a union of loose knots is not necessarily a loose link, a fact also true of loose links in overtwisted 3-manifolds.

In [8], Legendrian contact homology is defined (for a certain class of (Y, ξ)), which is a pseudo-holomorphic curve invariant of Legendrian knots. There it is shown that $LCH(s_{D^n}(L)) = 0$ for any knot L . Theorem 1.2 implies every loose knot is the D^n -stabilization of another knot, thus every loose Legendrian has trivial LCH . Suppose X is any exact symplectic filling of Y , and Γ is an exact Lagrangian with Legendrian boundary, L . Then Γ induces an augmentation of $LCH(L)$ [7]. The trivial algebra admits no augmentation, thus it follows that loose knots are not fillable by exact Lagrangians.

If a Weinstein manifold admits a handle decomposition so that the top dimensional handles are attached along loose knots, it is expected to inherit flexible properties. This topic will be explored in more depth in [4].

For tight contact structures on 3-dimensional manifolds, one could hope to find a similar flexible class of Legendrians. However, it is known that no suitable one exists: the standard unknot is the unique Legendrian knot in its formal isotopy class [12]. It is further shown there that the only topologically local structure of Legendrian curves is the number of stabilizations, and a result from [16] states that for any fixed k , there are formally isotopic knots which are distinct even after k stabilizations of any type. Together these results imply there is no flexible class of Legendrian 1-knots, which can be defined independent of the global topology of the knot.

Theorem 1.2 is for parametrized Legendrians, so it implies that loose knots have maximal $\pi_0 \text{Diff}(L)$ symmetry: any diffeomorphism of L fixing the classical invariants can be realized by an ambient contact isotopy. Of course any exotic diffeomorphism homotopic to the identity is an example of such a symmetry.

These facts raise questions about the complementary class of knots. Given L and (Y, ξ) , is there a non-loose knot in every formal isotopy class? Are there non-loose knots which are not fillable by exact Legendrians? Are there non-loose knots which do not admit maximal $\pi_0 \text{Diff}(L)$ symmetry? Many of these questions are open, even for spheres in \mathbb{R}_{std}^{2n+1} . See [8] for constructions of many non-loose spheres, and [9] for a construction of a non-loose Legendrian T^2 which admits no exact Lagrangian filling. The invariants defined in [8] and expanded upon in [7] are robust enough to detect all these phenomena.

Looking to results unique to high dimensional contact topology, we see that a “sufficiently thick” condition is often crucial. In [21] it is shown that any contact manifold containing a sufficiently thick Weinstein neighborhood of an overtwisted contact submanifold is not fillable by a (semi-positive) symplectic manifold. Note that all contact manifolds contain overtwisted submanifolds. In [13] it is shown that whenever $r < R < 1$, there is a contact isotopy of $(\mathbb{R}^{2n} \times S^1, \ker(d\theta - \sum_i y_i dx_i))$ which squeezes $B_R^{2n} \times S^1$ inside $B_r^{2n} \times S^1$. However, this is shown to be false when $r < 1 < R$. Though these results and Theorem 1.2 all seem intuitively similar, no concrete connections between these phenomena are presently understood.

APPENDIX A. FORMAL LEGENDRIAN ISOTOPY CLASSES IN \mathbb{R}_{std}^{2n+1}

In order for the main result of this paper to be useful in practice, we would like to have an explicit way to tell when two knots are formally isotopic. This is purely an issue about bundle theory and algebraic topology. The calculations are not particularly deep, but they are somewhat involved. First we define two invariants of formal Legendrian knots. Some of the details in calculation are left to the reader, they can also be found in [8].

Definition A.1. Let L be a formal Legendrian knot in (Y, ξ) . F_1 is a bundle map $TL \rightarrow \xi|_L$, so every fiber has Lagrangian image. The homotopy class of this map in the space of Lagrangian bundle monomorphisms is called the *rotation class* of L . We denote this class $r(L)$.

Immersed Legendrian knots satisfy an h-principle [18], and the rotation class classifies them up to regular Legendrian homotopy. If we have two Legendrian knots which are smoothly homotopic, we can compare their rotation classes as follows. A formal Legendrian defines an isomorphism $\xi|_L \cong TL \otimes \mathbb{C}$, therefore two formal Legendrians together define a difference element in $\text{Aut}_{\mathbb{C}}(TL \otimes \mathbb{C})$, also known as the gauge group of $\xi|_L$. Two Legendrians have the same rotation class if and only if this difference element is in the component of the identity. If $\xi|_L$ is trivial (which is always the case if $\xi|_Y$ is trivial) then $\text{Aut}_{\mathbb{C}}(\xi|_L) \cong \text{Map}(L, U_n)$, thus the difference class $r(L_0) - r(L_1)$ is an element of $K^1(L)$ in this case.

Definition A.2. Suppose n is odd, and let L be a formal Legendrian knot in (Y, ξ) . Assume L is orientable and nullhomologous. Extend F_s to a path \tilde{F}_s in $\text{Aut}_{\mathbb{R}}(TY|_L)$. Let R be a vector field in $TY|_L$, positively transverse to ξ . Then $\tilde{F}_1^{-1}(R)$ is nowhere tangent to L . The linking number of the knot with the vector field does not depend on the choice of lifting \tilde{F}_s . This integer is called the *Thurston-Bennequin number* of L , denoted $tb(L)$.

Remark. When n is even, the definition makes sense but the invariant is uninteresting. In the example \mathbb{R}_{std}^{2n+1} , we can equivalently consider the signed count of self intersections in the Lagrangian projection (regardless of dimension). If n is odd, the intersection product is skew, and the order of the inputs are given by height.

For even n the intersection product is commutative, so all the data necessary to calculate $tb(L)$ is contained in the Lagrangian projection. Together with the Lagrangian neighborhood theorem, it follows that $tb(L) = \frac{1}{2}(-1)^{n/2+1}\chi(L)$ in this case.

Proposition A.3. *Let L be a Legendrian knot, and $M \subseteq D^n$. Then $r(s_M(L)) = r(L)$ always. When n is odd, $tb(s_M(L)) = tb(L) - 2\chi(M)$.*

Proof: L and $s_M(L)$ are Legendrian regular homotopic by the homotopy $\{z = t \cdot h(x)\}$ so the statement about rotation class is clear (see Definition 2.4 for notation). For tb , we count the self intersections in the Lagrangian projection. In the course of the homotopy, L intersects itself once for each Morse critical point of h . This corresponds to a sign change for the associated intersection in the Lagrangian projection, so tb changes by ± 2 for each Morse critical point. By explicit calculation we see that the sign corresponds to the parity of the Morse index, so the total change is $2\chi(M)$. \square

We now state a classification of formal Legendrian isotopy classes, assuming our ambient manifold is $(\mathbb{R}^{2n+1}, \xi_{std})$. This tells us that all embeddings of L are smoothly isotopic when $n > 1$ [19]. Similar calculations can be done for any (Y, ξ) , but there are smooth obstructions and the bundle theory becomes more difficult.

Theorem A.4. *We describe Legendrian knots up to formal isotopy in \mathbb{R}_{std}^{2n+1} .*

- (a) *Suppose $n > 1$ is odd. If two Legendrian knots have the same Thurston-Bennequin number and rotation class, then they are formally Legendrian isotopic.*
- (b) *If two Legendrian surfaces in \mathbb{R}_{std}^5 have the same rotation class, they are formally Legendrian isotopic.*
- (c) *Suppose $n > 2$ is even. Then for each rotation class there are at most two formal Legendrian isotopy classes. If L is simply connected, there are exactly two.*

Remarks. It is unknown to the author if there exists a calculable invariant in \mathbb{Z}_2 which distinguishes the formal isotopy classes in case (c). Below it is defined as an invariant associated to a smooth isotopy between two Legendrian knots, which is why the $\pi_1 L = 0$ assumption is needed. The invariant in question should be a “Thurston-Bennequin-Kervaire semicharacteristic”, see [1].

Given any two odd dimensional knots with $r(L_1) = r(L_2)$, $tb(L_1) - tb(L_2)$ must be even. In fact for $n > 3$, the parity of $tb(L)$ is determined only by the topology of L . For example, $tb(S^n)$ is odd for any Legendrian sphere. To show this, first take the Lagrangian projection of L , which is an exact Lagrangian immersion in \mathbb{R}_{std}^{2n} . Notice the parity of $tb(L)$ is equal to the mod 2 count of self interesections of this Lagrangian immersion. In fact, this is an invariant of *smooth* immersions in \mathbb{R}_{std}^{2n} , up to regular homotopy. Both smooth and Lagrangian immersions satisfy h-principles [18], thus the existence of Lagrangian immersions of a given smooth regular homotopy class is governed by the inclusion map $\pi_n U_n \rightarrow \pi_n V_{2n,n}$. For n odd this is a map $\mathbb{Z} \rightarrow \mathbb{Z}_2$, and (a stable shift of) Lemma A.6 implies this is the

zero map except when $n = 1, 3$.

Proof: We assume some basic facts about frame bundles, see [2], [22]. Given two Legendrian knots construct a smooth isotopy L_t between them, this defines a path $\beta_t : L \rightarrow V_{2n+1,n}$ so that β_0 is a constant map (here $V_{2n+1,n}$ is the Stiefel manifold of n -frames in \mathbb{R}^{2n+1}). L need not admit a global parallelization, since here β_t compares dL_t to dL_0 at each point of L , and this difference does not depend on a choice of framing at that point. Said differently, maps $L \rightarrow Gr_n(\mathbb{R}^{2n+1}_{std})$ lifting the isotopy L_t can be identified with $\text{Map}(L, V_{2n+1,n})$ by choosing a connection on the tautological bundle over $Gr_n(\mathbb{R}^{2n+1})$. Inside $V_{2n+1,n}$, identify U_n as the subset of Legendrian frames. (Though “which frames are Legendrian” depends on the point in \mathbb{R}^{2n+1} , these inclusions are all homotopy equivalent to the inclusion $U_n \subseteq O_{2n} \subseteq O_{2n+1} \rightarrow V_{2n+1,n}$.) β_1 has image inside of U_n since L_1 is Legendrian, and so β_t defines an element $\beta \in \pi_1(\text{Map}(L, V_{2n+1,n}), \text{Map}(L, U_n))$. Notice that $\text{Map}(L, V_{2n+1,n})$ is connected since $V_{2n+1,n}$ is n -connected.

Our smooth isotopy can be made into a formal Legendrian isotopy exactly when $\beta = 0$. Conversely, given any $\beta \in \pi_1(\text{Map}(L, V_{2n+1,n}), \text{Map}(L, U_n))$ and a Legendrian knot L_0 , we can define a formal Legendrian knot $(f, F_s) = (L_0, \beta_s)$. If L_1 is a Legendrian realizing this formal Legendrian (which exists by Proposition 7.2), then the obstruction associated to the smooth isotopy between L_0 and L_1 is β .

In the long exact sequence for the pair, notice $\partial_*\beta = r(L_0) - r(L_1) \in \pi_0 \text{Map}(L, U_n)$. Thus under the assumption $r(L_0) = r(L_1)$ we can lift β to $\tilde{\beta} \in \pi_1 \text{Map}(L, V_{2n+1,n})$. We pause to prove some lemmas concerning the homotopy groups of frame bundles.

Lemma A.5. *Consider the fibration $O_{n+1} \rightarrow O_{2n+1} \rightarrow V_{2n+1,n}$. In the homotopy long exact sequence, the map $\pi_{n+1}V_{2n+1,n} \rightarrow \pi_n O_{n+1}$ is injective, except for $n = 2, 6$. For these two values, $\pi_n O_{n+1}$ is trivial.*

Proof: First, consider the case where n is odd. The kernel of our map is the image of the group $\pi_{n+1}O_{2n+1}$. By Bott periodicity, this group is finite. But $\pi_{n+1}V_{2n+1,n} \cong \mathbb{Z}$, so the image must be trivial.

Next, consider the case where n is even, and not equal to 2 or 6. Consider the map $\pi_n O_{n+1} \rightarrow \pi_n O_{2n+1}$. The first group classifies $(n+1)$ -vector bundles on S^{n+1} , whereas the second group classifies stable bundles. Since TS^{n+1} is non-trivial, but stably trivial [3], we know this map must have non-zero kernel. So $\pi_{n+1}V_{2n+1,n} \rightarrow \pi_n O_{n+1}$ has non-zero image. Since $\pi_{n+1}V_{2n+1,n} \cong \mathbb{Z}_2$, this implies the map is injective. \square

Lemma A.6. *For all $n > 2$, $\pi_{n+1}U_n \rightarrow \pi_{n+1}V_{2n+1,n}$ is the zero map. For $n = 2$, it is a surjection.*

Proof: Let $n \neq 2, 6$. Notice that the inclusion $U_n \subseteq V_{2n+1,n}$ factors through $U_n \subseteq O_{2n+1} \rightarrow V_{2n+1,n}$. By the previous lemma, the second map is trivial on π_{n+1} .

Case $n = 6$: Consider the map $\pi_{n+1}U_n \rightarrow \pi_{n+1}O_{2n+1}$. This is in the stable range, so we can look at the exact sequence

$$\pi_{n+1}U \rightarrow \pi_{n+1}O \rightarrow \pi_{n+1}(O/U) \rightarrow \pi_n U$$

By Bott periodicity, $\pi_n U \cong 0$, and $\pi_{n+1}(O/U) \cong \pi_{n+1}(\Omega O) \cong \mathbb{Z}_2$. It follows that the map $\pi_{n+1}U_n \rightarrow \pi_{n+1}O_{2n+1}$ is multiplication by 2, as a map $\mathbb{Z} \rightarrow \mathbb{Z}$. Therefore, the map $\pi_{n+1}U_n \rightarrow \pi_{n+1}V_{2n+1,n} \cong \mathbb{Z}_2$ is zero.

Case $n = 2$: Since $\pi_n O_{n+1} \cong 0$, we know $\pi_{n+1}O_{2n+1}$ surjects onto $\pi_{n+1}V_{2n+1,n}$. This, together with the fact that $\pi_{n+1}U_n \rightarrow \pi_{n+1}O_{2n+1}$ is an isomorphism, implies the result. \square

Lemma A.7. *Let n be odd. From the fibrations $O_{n+1} \rightarrow O_{2n+1} \rightarrow V_{2n+1,n}$ and $O_n \rightarrow O_{n+1} \rightarrow S^n$, form the composition map $tb : \pi_{n+1}V_{2n+1,n} \rightarrow \pi_n O_{n+1} \rightarrow \pi_n S^n$. Then tb is an injection, in fact, it is the map $\mathbb{Z} \mapsto 2\mathbb{Z}$.*

Proof: We know from Lemma A.5 that the first map is an injection, so the lemma is equivalent to “Is $\text{Im}(\pi_{n+1}V_{2n+1,n}) \cap \ker(\rightarrow \pi_n S^n)$ trivial in $\pi_n O_{n+1}$?” By the exact sequences, this is equivalent to the intersection $\ker(\rightarrow \pi_n O_{2n+1}) \cap \text{Im}(\pi_n O_n) \subseteq \pi_n O_{n+1}$. This statement is then equivalent to “Suppose ν is an $(n+1)$ -plane bundle on S^{n+1} which is both stably trivial and of zero euler class. Is ν trivial?”. But ν must be trivial; the tangent bundle of the sphere generates the group of stably trivial vector bundles over S^{n+1} , and it has nonzero euler class. The second statement follows since the euler class of this generator is 2. \square

Returning to the proof of the theorem, recall our isotopy is unobstructed if $\tilde{\beta} \in \pi_1 \text{Map}(L, V_{2n+1,n})$ is in the image of $\pi_1 \text{Map}(L, U_n)$. Take any degree one map $L \rightarrow S^n$. Since $V_{2n+1,n}$ is n -connected this map induces an isomorphism $\pi_1 \text{Map}(L, V_{2n+1,n}) \cong \pi_{n+1}V_{2n+1,n}$, identifying the image of $\pi_1 \text{Map}(L, U_n)$ with that of $\pi_{n+1}U_n$.

For part (b), $n = 2$: Lemma A.6 implies that that $\tilde{\beta}$ is in the image of $\pi_{n+1}U_n$, thus $\beta = 0$.

In part (a), n is odd. We claim $tb(\tilde{\beta}) = tb(L_0) - tb(L_1)$. Since Lemma A.7 says $tb : \pi_{n+1}V_{2n+1,n} \rightarrow \pi_n S^n$ is an injection and $tb(L_0) = tb(L_1)$ by hypothesis, this implies $\tilde{\beta} = 0$. Consider the geometric meaning of the maps in Lemma A.7. The first map to $\pi_n O_{n+1}$ can be interpreted as a difference class of the Legendrian framings of the normal bundle induced by the isotopy. The second map, induced by $O_{n+1} \rightarrow S^n$, is simply “pick one vector in the frame”, here we think of it as choosing the Reeb vector field. Thus $tb(\tilde{\beta})$ represents the difference class of the Reeb framings, which equals $tb(L_0) - tb(L_1)$.

For part (c), $n > 2$ is even. $\tilde{\beta} \in \pi_{n+1}V_{2n+1,n} \cong \mathbb{Z}_2$, which implies there are at most two formal Legendrian isotopy classes for the given rotation class. However $\tilde{\beta}$ is an invariant of a smooth isotopy: one can imagine a isotopy from a Legendrian to itself so that $\tilde{\beta} \neq 0$. If such a case exists there will only be one formal isotopy class

for the given rotation class. Under the assumption $\pi_1 L = 0$, the space of smooth embeddings $L \hookrightarrow \mathbb{R}_{std}^{2n+1}$ is simply connected [5] and thus this cannot occur. \square

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